大問2

(1) $S_n = \int_a^1 \left(\frac{1}{x^2}-1\right)^{\frac{n}{2}} dx$

$\left\{\left(\frac{1}{x^2}-1\right)^{\frac{n}{2}+1}\right\}' = \frac{n+2}{2}\left(\frac{1}{x^2}-1\right)^{\frac{n}{2}} \cdot \left(\frac{1}{x^2}-1\right)'$

$= \frac{n+2}{2}\left(\frac{1}{x^2}-1\right)^{\frac{n}{2}} \cdot \left(-\frac{2}{x^3}\right)$

$-x\left\{\left(\frac{1}{x^2}-1\right)^{\frac{n+2}{2}}\right\}' = \frac{n+2}{2}\left(\frac{1}{x^2}-1\right)^{\frac{n}{2}} \cdot \frac{2}{x^2}$

$-\int_a^1 x\left\{\left(\frac{1}{x^2}-1\right)^{\frac{n+2}{2}}\right\}' dx = (n+2)\int_a^1 \frac{1}{x^2}\left(\frac{1}{x^2}-1\right)^{\frac{n}{2}} dx$ ……①

$S_{n+2} = \int_a^1 \left(\frac{1}{x^2}-1\right)^{\frac{n+2}{2}}$

$= \int_a^1 \left(\frac{1}{x^2}-1\right)^{\frac{n}{2}} \cdot \left(\frac{1}{x^2}-1\right) dx$

$= \int_a^1 \frac{1}{x^2}\left(\frac{1}{x^2}-1\right)^{\frac{n}{2}} dx - \int_a^1 \left(\frac{1}{x^2}-1\right)^{\frac{n}{2}} dx$

$$S_{n+2} = \int_a^1 \frac{1}{x^2}\left(\frac{1}{x^2}-1\right)^{\frac{n}{2}}dx - S_n$$

…… ②

①, ② より

$$(n+2)\int_a^1 \frac{1}{x^2}\left(\frac{1}{x^2}-1\right)^{\frac{n}{2}}dx = -\int_a^1 x\left\{\left(\frac{1}{x^2}-1\right)^{\frac{n+2}{2}}\right\}'dx$$

$$= (n+2)S_{n+2} + (n+2)S_n$$

カ Ⓟ , キ Ⓟ

また、$S_{n+2} = \int_a^1 \left(\frac{1}{x^2}-1\right)^{\frac{n+2}{2}}dx$

$$= \int_a^1 (x)'\left(\frac{1}{x^2}-1\right)^{\frac{n+2}{2}}dx$$

$$= \left[x\left(\frac{1}{x^2}-1\right)^{\frac{n+2}{2}}\right]_a^1 - \int_a^1 x\left\{\left(\frac{1}{x^2}-1\right)^{\frac{n+2}{2}}\right\}'dx$$

$$S_{n+2} = -a\left(\frac{1}{a^2}-1\right)^{\frac{n+2}{2}} + 2S_{n+2} + 2S_n$$

$$(n+2)S_n + (n+1)S_{n+2} = a\left(\frac{1}{a^2}-1\right)^{\frac{n+2}{2}}$$

7 (f)

$\frac{2b+1}{2}$

(2) $$(n+2)S_n + nS_{n+1} + (n+1)S_{n+2}$$

$$= a\left(\frac{1}{a^2}-1\right)^{\frac{n+2}{2}} + nS_{n+1} > 0$$

$$-a\left(\frac{1}{a^2}-1\right)^{\frac{n+2}{2}} < nS_{n+1} \quad \cdots ③$$

また $$(n+3)S_{n+1} + (n+2)S_{n+3} = a\left(\frac{1}{a^2}-1\right)^{\frac{n+3}{2}}$$

$$(n+3)S_{n+1} = a\left(\frac{1}{a^2}-1\right)^{\frac{n+3}{2}} - (n+2)S_{n+3}$$

$$< a\left(\frac{1}{a^2}-1\right)^{\frac{n+3}{2}}$$

$$n S_{n+1} < a\left(\frac{1}{a^2}-1\right)^{\frac{n+3}{n}} - 3S_{n+1}$$

$$< a\left(\frac{1}{a^2}-1\right)^{\frac{n+3}{n}} \quad \cdots ④$$

従って

$$-a\left(\frac{1}{a^2}-1\right)^{\frac{n+2}{n}} < n S_{n+1} < a\left(\frac{1}{a^2}-1\right)^{\frac{n+3}{n}}$$

$$\lim_{n\to\infty}\left\{-a\left(\frac{1}{a^2}-1\right)^{\frac{n+2}{n}}\right\} < \lim_{n\to\infty} n S_{n+1} < \lim_{n\to\infty} a\left(\frac{1}{a^2}-1\right)^{\frac{n+2}{n}}$$

$$0 < \lim_{n\to\infty} n S_{n+1} < \Theta$$

$$\lim_{n\to\infty} n S_{n+1} = 0$$

ア ⓔ

(3)

$$(n+2)S_n + (n+1)S_{n+2} = a\left(\frac{1}{a^2}-1\right)^{\frac{n+2}{2}}$$

$$L_k = (2k+1)S_{2k-1} + 2k S_{2k+1}$$

$$R_k = a\left(\frac{1}{a^2}-1\right)^{\frac{2k+1}{2}}$$

$$(-1)^{k+1} L_k = (-1)^{k+1}(2k+1) S_{2k-1} + (-1)^{k+1}\cdot 2k S_{2k+1}$$

$$\sum_{k=1}^{N}(-1)^{k+1} L_k = \sum_{k=1}^{N}(-1)^{k+1}\cdot 2k S_{2k-1} + \sum_{k=1}^{N}(-1)^{k+1} S_{2k-1} + \sum_{k=1}^{N}(-1)^{k+1}\cdot 2k S_{2k+1}$$

ここで $\sum_{k=1}^{N}(-1)^{k+1} 2k S_{2k-1} + \sum_{k=1}^{N}(-1)^{k+1}\cdot 2k S_{2k+1}$ を計算する

$$\sum_{k=1}^{N}(-1)^{k+1} 2k S_{2k-1} = (-1)^2\cdot 2 S_1 + (-1)^3 2\cdot 2 S_3 + (-1)^4\cdot 2\cdot 3 S_5 + \cdots + (-1)^{N+1} 2\cdot N S_{N-1}$$

$$\sum_{k=1}^{N}(-1)^{k+1} 2k S_{2k+1} = (-1)^2\cdot 2 S_3 + (-1)^3\cdot 2\cdot 2 S_5 + (-1)^4\cdot 2\cdot 3 S_7 + (-1)^N\cdot 2(N-1) S_{N-1} + (-1)^{N+1}\cdot 2N S_{N+1}$$

$$2\rho_1 - 2\rho_3 + 2\rho_5 - 2\rho_7 + \cdots + (-1)^{N+1} \cdot 2\rho_{N-1}$$
$$(-1)^{N+1} \cdot 2N\rho_{N+1}$$

$$= 2\left(\rho_1 - \rho_3 + \rho_5 - \rho_7 + \cdots + (-1)^{N+1}\rho_{N-1}\right)$$
$$+ (-1)^{N+1} \cdot 2N\rho_{N+1}$$

よって
$$\sum_{k=1}^{N}(-1)^{k+1}\rho_{2k-1} + 2\sum_{k=1}^{N}(-1)^{k+1}\rho_{2k-1}$$
$$+ (-1)^{N+1} \cdot 2N\rho_{N+1}$$
$$= 3\sum_{k=1}^{N}(-1)^{k+1}\rho_{2k-1} + (-1)^{N+1} \cdot 2N\rho_{N+1}$$

= ⓗ + ⓓ //

$$3\sum_{k=1}^{N}(-1)^{k+1}\rho_{2k-1} = \sum_{k=1}^{N}(-1)^{k+1}R_k$$
$$- (-1)^{N+1} 2N\rho_{N+1}$$

$$\sum_{k=1}^{N}(-1)^{k+1}\rho_{2k-1} = \frac{1}{3}\left\{\sum_{k=1}^{N}(-1)^{k+1}R_k - (-1)^{N+1} \cdot 2N\rho_{N+1}\right\}$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} S_{2k-1} = \frac{1}{3}\lim_{N\to\infty}\left\{ \sum_{k=1}^{N} (-1)^{k+1} R_k \right.$$

$$\left. (-1)^{N+1}\cdot 2N S_{N+1} \right\}$$

$$= \frac{1}{3}\lim_{N\to\infty} \sum_{k=1}^{N} (-1)^{k+1} R_k$$

$$= -\frac{1}{3}\lim_{N\to\infty} \sum_{k=1}^{N} (-1)^{k+1}\cdot a\left(\frac{1}{a^2}-1\right)^{\frac{2k+1}{2}}$$

$$a\left(\frac{1}{a^2}-1\right)^{\frac{2k+1}{2}} = a\left(\frac{1}{a^2}-1\right)^{k+\frac{1}{2}}$$ より

$$-\frac{a}{3}\cdot\left(\frac{1}{a^2}-1\right)^{\frac{1}{2}}\lim_{N\to\infty}\sum_{k=1}^{N}(-1)^k\cdot\left(\frac{1}{a^2}-1\right)^k$$

$$= -\frac{a}{3}\left(\frac{1}{a^2}-1\right)^{\frac{1}{2}}\lim_{N\to\infty}\sum_{k=1}^{N}\left(\frac{a^2-1}{a^2}\right)^k$$

$$\lim_{N\to\infty}\sum_{k=1}^{N}\left(1-\frac{1}{a^2}\right)^k = \frac{\frac{a^2-1}{a^2}}{1-\frac{a^2-1}{a^2}}$$

$$= \frac{\frac{a^2-1}{a^2}}{\frac{a^2-a^2+1}{a^2}} = a^2-1$$

よって $-\frac{a}{3}\cdot\left(\frac{1-a^2}{a^2}\right)^{\frac{1}{2}}\cdot(a^2-1)$

$= \frac{(1-a^2)^{\frac{3}{2}}}{3}$

$=$ ㋒

//